

Cooperative Localization in WSNs: a Hybrid Convex/non-Convex Solution

Nicola Piovesan, Tomaso Erseghe

Abstract—We propose an efficient solution to peer-to-peer localization in a wireless sensor network which works in two stages. At the first stage the optimization problem is relaxed into a convex problem, given in the form recently proposed by Soares, Xavier, and Gomes. The convex problem is efficiently solved in a distributed way by an ADMM approach, which provides a significant improvement in speed with respect to the original solution. In the second stage, a soft transition to the original, non-convex, non relaxed formulation is applied in such a way to force the solution towards a local minimum. The algorithm is built in such a way to be fully distributed, and it is tested in meaningful situations, showing its effectiveness in localization accuracy and speed of convergence, as well as its inner robustness.

Index Terms—Alternating direction method of multipliers, Decentralized estimation, Distributed algorithms, Cooperative Localization, Maximum likelihood, Optimization methods, Wireless sensor networks.

I. INTRODUCTION

SMART SENSORS nowadays have grown rapidly thanks to the proliferation of micro electro-mechanical system (MEMS) technology and advances in radio frequency (RF) communications. This kind of sensors are the basic unit of a wireless sensor network (WSN), which are extending our “ability to monitor and control the physical world.” Sensor nodes can in fact “sense, measure and gather information from the environment and, based on some local decision process, they can transmit the sensed data to the user” [1]. They are characterized by their reduced dimension, limited processing and computing resources, and low costs. WSNs can be used either for monitoring applications (which include indoor/outdoor environmental monitoring, seismic monitoring, health monitoring), but also for tracking applications (of objects, animals, humans and vehicles). In this latter context, cooperative localization in WSNs is a task that has gained increasing interest, especially in indoor scenarios where satellite communications cannot be employed, but also for the inner capability of providing more accurate results. The possibility of efficiently locating the objects opens up a wide number of applications ranging from the industrial to the health-care environment, and it is also beneficial for an efficient management of the communication network itself [2]. However, it also calls for suitably fast and simple solutions.

In this scenario, the localization problem we aim at solving is one where a number, N , of nodes with ranging capability

wirelessly exchanges ranging measurements with its neighboring nodes, i.e., those inside a given communication radius. By letting \mathbf{p}_i be the position of the i th node, \mathcal{N}_i the set of neighbor nodes to which it communicates and for which a ranging measurement is available, and $r_{i,j}$ the noisy ranging measurement between node i and its neighbor $j \in \mathcal{N}_i$, then the optimization problem we wish to solve is of the form

$$\begin{aligned} \min \quad & \sum_{i \in \mathcal{N}, j \in \mathcal{N}_i} \frac{1}{2} (\|\mathbf{p}_i - \mathbf{p}_j\| - r_{i,j})^2 \\ \text{w.r.t. } & \mathbf{p}_i, i \in \mathcal{N} = \{1, \dots, N\} \\ \text{s.t. } & \mathbf{p}_k = \mathbf{a}_k, k \in \mathcal{A}. \end{aligned} \quad (1)$$

In the problem formulation we implicitly considered that the ranging measurement is corrupted by an additive white Gaussian noise (AWGN), and we also considered that a subset $\mathcal{A} \subset \mathcal{N}$ of the nodes are *anchor* nodes, i.e., nodes for which the exact position \mathbf{a}_i is known. In the intended scenario, only a very few nodes are anchors (e.g., the all-but-one-node are anchors is not the intentional scope of our investigation), which makes the problem highly non-convex, and in general difficult to solve.

The above problem has been previously considered, e.g., in [3]–[12]. We concentrate on the more relevant solutions. The approach considered in [7], [8], [10] uses a semi-definite programming (SDP) relaxation in order to map the non-convex original problem into a convex problem. Although the idea is potentially interesting, especially because a convergence guarantee can be obtained, the SDP relaxation implies a non trivial computational effort which makes it impractical in large networks. A valid alternative is provided by the convexification method proposed by Soares, Xavier, and Gomez in [11]. By exploiting the concept of convex envelope, the authors are able to identify an algorithm which is simple and scalable. Since all of the above mentioned solutions are expected to provide an identical performance, [11] should be considered the preferred convex relaxation approach. Its main drawback, however, is the lack of adherence to the original non-convex problem, which practically means that the solutions are not guaranteed to be global, nor local, minima. In this interpretation, working directly on the non-convex problem (1) might be a viable option. This option was investigated by Erseghe in [12], which proposes a solution based upon the alternating direction method of multipliers (ADMM) method, suitably modified in order to guarantee convergence to a local minimum also in the presence of harsh non-convexities. The plain ADMM solution is in fact known to converge only under convex problem formulations. The resulting algorithm is

The authors are with the Dipartimento di Ingegneria dell’Informazione, Università di Padova, Via G. Gradenigo 6/B, 35131 Padova, Italy. Contact author: Tomaso Erseghe, tel: +39 049 827 7656, fax: +39 049 827 7699, mailto: erseghe@dei.unipd.it

effective and implies a controlled computational burden. The main drawback with this latter method is, however, a general difficulty to ensure convergence to a good local minimum, especially in a worse case situation where the starting point is very far from optimum, and the problem is highly non-convex.

In this paper, we aim at bringing together the positive aspects of [11] and [12], by proposing an hybrid solution to the WSN cooperative localization problem (1). The leading idea is to exploit the convex relaxation method introduced in [11] in order to identify a starting solution which is then refined – and guaranteed to be at least a local optimum – by using the ideas developed in [12]. By using a relaxation we also expect to obtain a faster convergence rate than with the standard non-convex approach, the reason simply being the higher level of coordination ensured by removing non-convexities. The transition between the two approaches is meant to be smooth, and to be implementable in a fully distributed fashion. To this aim, the ADMM method is used to solve both the original problem as well as its convex counterpart. More precisely, the application of ADMM to the convex relaxation problem of [11] will be shown to provide a significant improvement over the parallel Nesterov's method originally proposed by the authors, that is, a much faster convergence rate. The activation of the non-convex function under local convergence of the relaxation will further ensure convergence to a local optimum. This gives a gain in localization accuracy over [11], which will be shown to be rather significant. The localization accuracy of the proposed solution is equivalent to the one that can be obtained by [12], but with two relevant enhancements: a faster convergence speed and, even more importantly, a strong resilience with respect to the parameters choice which improves the algorithm robustness.

The paper is organized as follows. The convex relaxation is formalized in Section II, the problem is put in a networked form which is suitable for being implemented in a distributed fashion in Section III, the overall distributed algorithm is presented in Section IV, and its computational complexity is discussed in Section V. Performance is assessed in Section VI, where a comparison with state-of-the-art solutions is also given. Section VII concludes the paper.

II. CONVEX RELAXATION

As we discussed, the constituent functions of (1) are of the form

$$F_i(\mathbf{x}_i) = \sum_{j \in \mathcal{N}_i} \frac{1}{2} (\|\mathbf{p}_i - \mathbf{p}_j\| - r_{i,j})^2, \quad (2)$$

where $\mathbf{x}_i = \{\mathbf{p}_j\}_{j \in \{i\} \cup \mathcal{N}_i}$ collects copies of the position of node i together with the positions of its neighbors. Note that, unlike [11], the approach we are using in (1) does not involve two different notations for anchor and non-anchor nodes, and therefore it compacts and simplifies exposition.

Now, according to [11], the contribution

$$f(\mathbf{z}, r) = \frac{1}{2} (\|\mathbf{z}\| - r)^2 \quad (3)$$

can be equivalently written in the form

$$f(\mathbf{z}, r) = \min_{\mathbf{y}: \|\mathbf{y}\|=r} \frac{1}{2} \|\mathbf{z} - \mathbf{y}\|^2. \quad (4)$$

This provides the convex envelope through a simple relaxation of the non-convex constraint $\|\mathbf{y}\| = r$ into the convex constraint $\|\mathbf{y}\| \leq r$. To better evidence the convexity property, we can equivalently write $\|\mathbf{y}\|^2 \leq r^2$. We therefore obtain a convex relaxation based upon contributions of the form

$$\tilde{f}(\mathbf{z}, r) = \min_{\mathbf{y}: \|\mathbf{y}\| \leq r} \frac{1}{2} \|\mathbf{z} - \mathbf{y}\|^2, \quad (5)$$

the convexified counterpart to (2) being

$$\tilde{F}_i(\mathbf{x}_i) = \sum_{j \in \mathcal{N}_i} \tilde{f}(\mathbf{p}_i - \mathbf{p}_j, r_{i,j}). \quad (6)$$

We observe that \tilde{f} can be given in an explicit form, which reveals the simplicity of the convex relaxation, and which was not evidenced in [11]. Now, the function in (5) reaches its minimum when $\mathbf{y} = \mathbf{z}$. This solution is not always licit but we can say that the optimum solution corresponds to the closest allowed point \mathbf{y} to \mathbf{z} . We can therefore distinguish between two cases:

- 1) If $\|\mathbf{z}\| \leq r$, then \mathbf{z} is a member of the set from which we can select \mathbf{y} , and the value that minimizes the function is simply $\mathbf{y} = \mathbf{z}$. In this case, the function value is simply $\tilde{f}(\mathbf{z}, r) = 0$.
- 2) If, instead, $\|\mathbf{z}\| > r$, then the value of \mathbf{y} that minimizes (5) is on the boundary of the circle $\|\mathbf{y}\| = r$ and, more specifically, it corresponds to the intersection between the circle and the line that connects the point \mathbf{z} to the origin. We therefore have $\mathbf{y} = r\mathbf{z}/\|\mathbf{z}\|$, and an outcome equal to $\tilde{f}(\mathbf{z}, r) = \frac{1}{2} (\|\mathbf{z}\| - r)^2$.

To summarize these results, we can write

$$\tilde{f}(\mathbf{z}, r) = g(\|\mathbf{z}\| - r), \quad (7)$$

where

$$g(x) = \frac{1}{2} x^2 \cdot 1(x), \quad (8)$$

and where $1(x)$ is the unit step function, providing 1 for $x \geq 0$ and 0 for $x < 0$. Note a close relation between (4) and (7), the latter simply setting to zero the result when $\|\mathbf{z}\| < r$. An illustration of the effect given by the convex approximation can be found in [11, Fig. 1]. Note also that g has first and second derivatives

$$\begin{aligned} g'(x) &= x \cdot 1(x) = [x]^+ \\ g''(x) &= 1(x) \end{aligned} \quad (9)$$

which will be useful later on in order to identify gradients and Hessians.

III. PROBLEM FORMALIZATION

In order to approach the solution of (1), or of its convex counterpart replacing (2) with (6), the decomposition and coordination method of [12] is employed. In this context, problem (1) is put in an equivalent form where variables \mathbf{p}_i are duplicated in such a way that the generic node i owns its copy of variables $\mathbf{x}_i = \{\mathbf{p}_j\}_{j \in \mathcal{N}_i \cup \{i\}}$. Specifically, problem (1) assumes the form

$$\begin{aligned} &\min F(\mathbf{x}) \\ &\text{w.r.t. } \mathbf{x} \in \mathcal{X}, \mathbf{z} \in \mathcal{Z} \\ &\text{s.t. } \mathbf{A}\mathbf{x} = \mathbf{z}, \end{aligned} \quad (10)$$

where $\mathbf{x} = [\mathbf{x}_1, \dots, \mathbf{x}_N]$, with $\mathbf{x}_i = [\mathbf{x}_{i,j}]_{j \in \mathcal{N}_i \cup \{i\}}$ collecting in its entries the replicas of the position of node i , namely, $\mathbf{x}_{i,i} = \mathbf{p}_i$, and those of its neighbors $\mathbf{x}_{i,j} = \mathbf{p}_j$, $j \in \mathcal{N}_i$. If we denote with n the coordinate dimension, namely, $n = 2$ for 2D localization, and $n = 3$ for 3D localization, then \mathbf{x}_i has length $n(1 + N_i)$, with N_i the cardinality of \mathcal{N}_i , i.e., the number of neighbors of node i . The target function in (10) is the separable function

$$F(\mathbf{x}) = \sum_{i \in \mathcal{N}} F_i^\bullet(\mathbf{x}_i), \quad (11)$$

with $F_i^\bullet = F_i$ as defined in (2) if the original non-convex formulation is used, and $F_i^\bullet = \tilde{F}_i$ as defined by (6) if the convex relaxation is used. Moreover, set \mathcal{X} assumes the cartesian form $\mathcal{X} = \mathcal{X}_1 \times \dots \times \mathcal{X}_N$, where each \mathcal{X}_i is itself separable in the form

$$\mathcal{X}_i = \mathcal{R}_i \times \left\{ \bigotimes_{j \in \mathcal{N}_i} \mathcal{R}_j \right\}, \quad \mathcal{R}_i = \begin{cases} \mathcal{R} & i \notin \mathcal{A} \\ \{\mathbf{a}_i\} & i \in \mathcal{A}, \end{cases} \quad (12)$$

with \mathcal{R} any (bounded) region containing the nodes positions.

The equivalence between replicas of the same position is jointly ensured in (10) by the constraint $\mathbf{A}\mathbf{x} = \mathbf{z}$ and by the fact that $\mathbf{z} \in \mathcal{Z}$. Specifically, the linear constraint is locally given in the form¹

$$\mathbf{A}_i \mathbf{x}_i = \mathbf{z}_i = \begin{bmatrix} \mathbf{z}_i^- \\ \mathbf{z}_i^+ \end{bmatrix} \quad (13)$$

with

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{1}_{N_i} & -\mathbf{I}_{N_i} \\ \mathbf{1}_{N_i} & \mathbf{I}_{N_i} \end{bmatrix} \otimes \mathbf{I}_n, \quad (14)$$

where \otimes is the Kronecker product, $\mathbf{1}_k$ denotes a column vector of length k with all its entries set to 1, \mathbf{I}_k denotes the identity matrix of order k , and n is the coordinate dimension. For consistency, in the above we assumed that $\mathbf{z} = [\mathbf{z}_1, \dots, \mathbf{z}_N]$ is the collection of local contributions, and that $\mathbf{z}_i^- = [\mathbf{z}_{i,j}^-]_{j \in \mathcal{N}_i}$ and $\mathbf{z}_i^+ = [\mathbf{z}_{i,j}^+]_{j \in \mathcal{N}_i}$. As a consequence, the length of both \mathbf{z}_i^- and \mathbf{z}_i^+ is nN_i . It also is

$$\mathbf{A} = \text{diag}(\mathbf{A}_i, i \in \mathcal{N}). \quad (15)$$

Note that (13) separately identifies differences of the form $\mathbf{z}_{i,j}^- = \mathbf{p}_i - \mathbf{p}_j$, which are the ones effectively used in the target function (2), and contributions of the form $\mathbf{z}_{i,j}^+ = \mathbf{p}_i + \mathbf{p}_j$. These are made consistent throughout the network by forcing \mathbf{z} to belong to the linear space

$$\mathcal{Z} = \left\{ \mathbf{z} \mid \mathbf{z}_{i,j}^- = -\mathbf{z}_{j,i}^-, \mathbf{z}_{i,j}^+ = \mathbf{z}_{j,i}^+, \forall i \in \mathcal{N}, j \in \mathcal{N}_i \right\}. \quad (16)$$

The approach described so far is redundant in that it identifies $2nN_i$ constraints, $\mathbf{A}_i \mathbf{x}_i = \mathbf{z}_i$, for $n(1 + N_i)$ scalar variables, \mathbf{x}_i . However, a number of reasons make it desirable. It is in fact particularly well suited for distributed implementation, and, more importantly, it provides an advantage in terms of convergence speed. The reason for the latter originates from the use of variables $\mathbf{z}_{i,j}^- = \mathbf{p}_i - \mathbf{p}_j$ which allow to treat separately the convergence on relative positions $\mathbf{z}_{i,j}^-$

(which set the target value), from the convergence with respect to absolute positions $\mathbf{z}_{i,j}^+$ (which set the final localization outcome). Further details on this idea can be found by the interested reader in [12], [13].

IV. DISTRIBUTED ALGORITHM

A distributed algorithm can be obtained by applying the ADMM concept to (10). The ADMM is a simple but powerful algorithm that solves optimization problems decomposing them into smaller local sub-problems, which are easier to handle. The solutions to these local subproblems are coordinated in order to find the solution to a global problem. This algorithm is well suited for distributed optimization and in the latest years it found several applications in different areas [14], [15].

The ADMM method we use is taken from [12] and, in the present context, provides Algorithm 1. Specifically, in Algorithm 1 saddle points of the augmented Lagrangian

$$L(\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{c}) = \sum_{i \in \mathcal{N}} F_i^\bullet(\mathbf{x}_i) + \langle \boldsymbol{\lambda}_i, \mathbf{A}_i \mathbf{x}_i - \mathbf{z}_i \rangle + \frac{1}{2} c_i \|\mathbf{A}_i \mathbf{x}_i - \mathbf{z}_i\|^2 \quad (23)$$

are searched for by an alternate search that separately optimizes for each of the variables \mathbf{x} , \mathbf{z} , and $\boldsymbol{\lambda}$, where:

- 1) The optimization with respect to \mathbf{x} is obtained in line 8.
- 2) The optimization with respect to \mathbf{z} is obtained in lines 10-13 through a local exchange of information. Note that messages \mathbf{m}_i have the same structure of \mathbf{z}_i , that is $\mathbf{m}_i = [\mathbf{m}_i^-, \mathbf{m}_i^+]$.
- 3) The update of $\boldsymbol{\lambda}$ is obtained in line 15, where $\mathcal{P}_{\lambda_{\max}}$ performs a clipping of vector entries in the range $[-\lambda_{\max}; \lambda_{\max}]$, but any other clipping method can be used.

In line 19, if the non-convex formalization is used, the procedure updates the penalty parameters \mathbf{c} under two different conditions, namely: 1) if the primal gap $\mathbf{A}_i \mathbf{x}_i - \mathbf{z}_i$ does not decrease sufficiently, where the chosen measure corresponds to an infinity norm criterion – i.e., maximum value–; and 2) if the penalty parameters of neighbors have been previously increased. Note that the parameters used in (22) must satisfy $\delta_c > 1$ and $0 < \theta_c < 1$, but a reliable algorithm is obtained only with $\delta_c \gtrsim 1$ and $\theta_c \lesssim 1$.

A smooth transition between the convex relaxation employing functions \tilde{F}_i , and the non-convex original formulation employing functions F_i , is simply managed by starting from a non-convex formulation on every node (line 3), and by locally activating the non-convex functions as soon as the local primal gap exceeds a given threshold (lines 21-22). If constant τ_c is chosen sufficiently small, then the non-convex formalization is activated only after convergence is reached on the convexified problem.

To summarize, the parameters used in Algorithm 1 are:

- 1) The value ϵ_c for penalty parameters to be used with the convex formalization (line 6), and the initial value ζ_c for penalty parameters to be used with the non-convex formalization (line 23). These are the most relevant parameters that set the convergence speed and that must be wisely chosen according to the network

¹Note in the comparison with [12] that in (13)-(14) parameters ϵ and ζ have been dropped, i.e., they have been set to value 1, since we verified that this simplification does not affects the final algorithm performance.

Algorithm 1: Cooperative algorithm at node i .

```

1 for iteration counter  $t$  ranging from 0 to  $\infty$  do
2   if  $t = 0$  then
3     Start from the convexified problem and set
        $F_i^\bullet = \tilde{F}_i$  as in (6)
4     Initialize local positions  $\mathbf{x}_i = \mathbf{x}_i^0$ 
5     Initialize Lagrange multipliers  $\boldsymbol{\lambda}_i = \mathbf{0}$ 
6     Initialize penalty parameters  $c_i = \epsilon_c$ 
7   else
8     Update local positions  $\mathbf{x}_i$  via
        $\mathbf{x}_i \in \underset{\mathbf{x}_i \in \mathcal{X}_i}{\operatorname{argmin}} F_i^\bullet(\mathbf{x}_i) + \frac{1}{2}c_i(\mathbf{x}_i - \mathbf{y}_i)^T \mathbf{D}_i(\mathbf{x}_i - \mathbf{y}_i)$ 
       with  $\mathbf{y}_i = \mathbf{D}_i^{-1} \mathbf{A}_i^T(\mathbf{z}_i - \boldsymbol{\lambda}_i/c_i)$  and
        $\mathbf{D}_i = \mathbf{A}_i^T \mathbf{A}_i = \begin{bmatrix} 2N_i & \\ & 2I_{N_i} \end{bmatrix} \otimes \mathbf{I}_n$ 
9   end
10  Build messages  $\mathbf{m}_i = \mathbf{A}_i \mathbf{x}_i + \boldsymbol{\lambda}_i/c_i$ 
11   $\Rightarrow$  Broadcast values  $c_i, \mathbf{m}_{i,j}^+, \mathbf{m}_{i,j}^-$  to region  $j \in \mathcal{N}_i$ 
12   $\Leftarrow$  Receive values  $c_j, \mathbf{m}_{j,i}^+, \mathbf{m}_{j,i}^-$  from region  $j \in \mathcal{N}_i$ 
13  Extract the local projection on  $\mathcal{Z}$  by
        $\mathbf{z}_{i,j}^- = \frac{1}{2}(\mathbf{m}_{i,j}^- - \mathbf{m}_{j,i}^-)$ 
        $\mathbf{z}_{i,j}^+ = \frac{1}{2}(\mathbf{m}_{i,j}^+ + \mathbf{m}_{j,i}^+)$ 
14  if  $t > 0$  then
15    Update Lagrange multipliers
        $\boldsymbol{\lambda}_i = \mathcal{P}_{\lambda_{\max}}[\boldsymbol{\lambda}_i + c_i(\mathbf{A}_i \mathbf{x}_i - \mathbf{z}_i)]$ 
16  end
17  Evaluate the current primal gap
        $\Gamma_i = \|\mathbf{A}_i \mathbf{x}_i - \mathbf{z}_i\|_\infty$ 
18  if functionals are non-convex  $F_i^\bullet = F_i$  then
19    Update penalty parameters
        $c_i = \left( \max_{j \in \mathcal{N}_i \cup \{i\}} c_j \right) \cdot \begin{cases} 1 & , \Gamma_i \leq \theta_c \Gamma_{i,\text{old}} \\ \delta_c & , \text{otherwise} \end{cases}$ 
20  else
21    if primal gap is sufficiently small  $\Gamma_i < \tau_c$  then
22      Activate the non-convex problem by setting
         $F_i^\bullet = F_i$  as in (2)
23      Re-initialize penalty parameters  $c_i = \zeta_c$ 
24    end
25  end
26 end

```

characteristics. These values are kept separate since the non-convex functions F_i imply larger function values (e.g., see the pictorial representation in later Fig. 1), hence larger values of c_i to correctly balance function value and equality constraint in (17). For this reason we must also set $\zeta_c \gg \epsilon_c$. As detailed in [12], small variations in these values do not affect performance, provided that they are chosen in the correct range.

- 2) The clipping range λ_{\max} for Lagrange multipliers (line 15). The parameter should be chosen sufficiently high, in order to prevent unwanted clipping actions. The standard choice, which will be used later on in the numerical simulations section, is $\lambda_{\max} = 10^3$.
- 3) The update parameters δ_c and θ_c for penalty parameters (line 19). Standard choices, which will be used later on in the numerical simulations section, are $\delta_c = 1.01$ and $\theta_c = 0.98$.
- 4) The threshold τ_c for activating the original non-convex problem (line 21). This is another very relevant parameter that must be adequately chosen in dependence of the considered network in order to speed up convergence.

As proved in [12], which, in turn, derives from the ideas developed in [16], [17] about practical Lagrange methods, the fact that we bound both primal variables, \mathbf{x} and \mathbf{z} , as well as Lagrange multipliers, $\boldsymbol{\lambda}$, ensures that Algorithm 1 will converge. If the considered functions F_i were convex, then the limit point of Algorithm 1 would identify a global minimum. In the non-convex scenario, however, the algorithm may find a local, rather than a global, minimum.

V. COMPUTATIONAL COMPLEXITY CONSIDERATIONS AND FURTHER INSIGHTS

We observe that Algorithm 1 involves very simple operations, except for the local update (17) which corresponds to an optimization problem of order $n(1 + N_i)$. The problem can be approached via standard optimization techniques relying on gradients and Hessians, which can be compactly expressed in the form

$$\begin{aligned} \nabla F_i^\bullet(\mathbf{x}_i) &= \begin{bmatrix} \sum_{j \in \mathcal{N}_i} \mathbf{A}_{i,j} \\ -[\mathbf{A}_{i,j}]_{j \in \mathcal{N}_i} \end{bmatrix} \\ \nabla^2 F_i^\bullet(\mathbf{x}_i) &= \begin{bmatrix} \sum_{j \in \mathcal{N}_i} \mathbf{B}_{i,j} & -[\mathbf{B}_{i,j}]_{j \in \mathcal{N}_i}^T \\ -[\mathbf{B}_{i,j}]_{j \in \mathcal{N}_i} & \operatorname{diag}([\mathbf{B}_{i,j}]_{j \in \mathcal{N}_i}^T) \end{bmatrix}, \end{aligned} \quad (24)$$

with

$$\begin{aligned} \mathbf{A}_{i,j} &= \mathbf{A}(\mathbf{x}_{i,i} - \mathbf{x}_{i,j}, r_{i,j}) \\ \mathbf{B}_{i,j} &= \mathbf{B}(\mathbf{x}_{i,i} - \mathbf{x}_{i,j}, r_{i,j}), \end{aligned} \quad (25)$$

and where we used

$$\begin{aligned} A(\mathbf{z}, r) &= \frac{\mathbf{z}}{\|\mathbf{z}\|} [\|\mathbf{z}\| - r]^\bullet \\ B(\mathbf{z}, r) &= \frac{\mathbf{I}_n}{\|\mathbf{z}\|} [\|\mathbf{z}\| - r]^\bullet + \frac{\mathbf{z}\mathbf{z}^T}{\|\mathbf{z}\|^3} r \cdot 1^\bullet(\|\mathbf{z}\| - r). \end{aligned} \quad (26)$$

with

$$[x]^\bullet = x \cdot 1^\bullet(x), \quad 1^\bullet(x) = \begin{cases} 1 & \text{if } F_i \\ 1(x) & \text{if } \tilde{F}_i \end{cases} \quad (27)$$

to take into account both the convexified as well as the non-convex case.

However, when region \mathcal{R} is sufficiently large and ranging measurements are sufficiently reliable that we can drop the constraint given by \mathcal{R} , then the minimization problem (17) entails a simplified version. This possibility is a powerful result

that was not discussed in [12], and which we now separately address in case node i is an anchor and in case node i is not an anchor. Processing on anchors is a result which is preliminary to the problem simplification in the non-anchor case, and is therefore presented first.

A. Anchor nodes, $i \in \mathcal{A}$

For anchor nodes, due to the constraint $\mathbf{x}_{i,i} = \mathbf{a}_i$, problem (17) becomes separable, that is, it reduces to the parallel of N_i problems of the form

$$\mathbf{x}_{i,j} \in \operatorname{argmin}_{\mathbf{x} \in \mathcal{R}} \frac{1}{2}(\|\mathbf{x} - \mathbf{a}_i\| - r_{i,j})^2 + c_i \|\mathbf{x} - \mathbf{y}_{i,j}\|^2 \quad (28)$$

for $j \in \mathcal{N}_i$. When dealing with the original non-convex formulation (6), a closed-form solution to (28) can be easily derived from the zero-gradient condition (see the first of (24))

$$2c_i(\mathbf{x}_{i,j} - \mathbf{y}_{i,j}) = (\mathbf{a}_i - \mathbf{x}_{i,j}) \left(1 - \frac{r_{i,j}}{\|\mathbf{a}_i - \mathbf{x}_{i,j}\|}\right). \quad (29)$$

By setting $\mathbf{x}_{i,j} = \mathbf{a}_i + \alpha \mathbf{u}$ with $\|\mathbf{u}\| = 1$ and $\alpha > 0$, the condition (29) turns into

$$\mathbf{u}(\alpha(1 + 2c_i) - r_{i,j}) = 2c_i(\mathbf{y}_{i,j} - \mathbf{a}_i), \quad (30)$$

providing

$$\mathbf{u} = \frac{\mathbf{y}_{i,j} - \mathbf{a}_i}{\|\mathbf{y}_{i,j} - \mathbf{a}_i\|}, \quad \alpha = \frac{r_{i,j} + 2c_i\|\mathbf{y}_{i,j} - \mathbf{a}_i\|}{1 + 2c_i}. \quad (31)$$

For the convexified version (6), constraint (29) turns into

$$2c_i(\mathbf{x}_{i,j} - \mathbf{y}_{i,j}) = (\mathbf{a}_i - \mathbf{x}_{i,j}) \left[1 - \frac{r_{i,j}}{\|\mathbf{a}_i - \mathbf{x}_{i,j}\|}\right]^+, \quad (32)$$

so that the counterpart to (31) holds with α defined by

$$2c_i\alpha + [\alpha - r_{i,j}]^+ = 2c_i\|\mathbf{y}_{i,j} - \mathbf{a}_i\|, \quad (33)$$

which ensures

$$\alpha = \begin{cases} (31) & \text{if } \|\mathbf{y}_{i,j} - \mathbf{a}_i\| > r_{i,j} \\ \|\mathbf{y}_{i,j} - \mathbf{a}_i\| & \text{otherwise.} \end{cases} \quad (34)$$

By putting the above all together, we have

$$\mathbf{x}_{i,j} = \begin{cases} \mathbf{a}_i + \frac{r_{i,j} + 2c_i\|\mathbf{y}_{i,j} - \mathbf{a}_i\|}{1 + 2c_i} \frac{\mathbf{y}_{i,j} - \mathbf{a}_i}{\|\mathbf{y}_{i,j} - \mathbf{a}_i\|} & \text{if } F_i \text{ is used, or} \\ & \text{if } \tilde{F}_i \text{ is used and } \|\mathbf{y}_{i,j} - \mathbf{a}_i\| > r_{i,j} \\ \mathbf{y}_{i,j} & \text{otherwise} \end{cases} \quad (35)$$

which covers both the convexified case, (2), and the non-convex case, (6). A graphical interpretation of the result is given in Fig. 1.

Observe that, solving the local problem in anchor nodes requires very simple operations. Furthermore, the computational burden carried by (35) can be transferred from the anchor node to its neighbor nodes. Note in fact that the average action of the product \mathbf{A}_i^T in the definition of \mathbf{y}_i , in line 18 of Algorithm 1, is not needed since $\mathbf{y}_{i,i}$ is not used. Such a transfer is a reasonable choice whenever the anchor is connected to a large number of nodes, in which case a large overhead in communication is avoided.

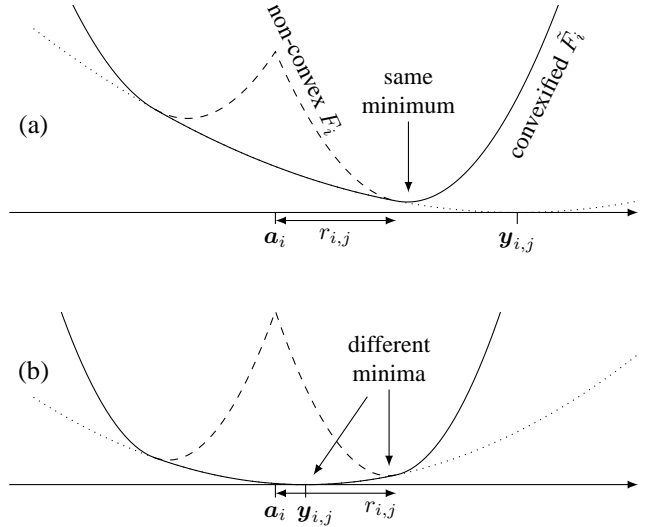


Fig. 1. Anchor node processing: identification of the optimal point for (a) $\|\mathbf{y}_{i,j} - \mathbf{a}_i\| > r_{i,j}$ and (b) $\|\mathbf{y}_{i,j} - \mathbf{a}_i\| < r_{i,j}$.

B. Nodes which are not anchors, $i \notin \mathcal{A}$

For nodes which are not anchors, the result given by (35) can be exploited to simplify the complexity of the problem from order $n(1 + N_i)$ to order n . In fact, for a fixed choice of $\mathbf{x}_{i,i}$ the local solution for $\mathbf{x}_{i,j}$ can be obtained from (35) by simply replacing \mathbf{a}_i with $\mathbf{x}_{i,i}$. This ensures that

$$\mathbf{x}_{i,j} = \mathbf{y}_{i,j} + \frac{\mathbf{x}_{i,i} - \mathbf{y}_{i,j}}{\|\mathbf{x}_{i,i} - \mathbf{y}_{i,j}\|} \cdot \frac{[\|\mathbf{x}_{i,i} - \mathbf{y}_{i,j}\| - r_{i,j}]^+}{1 + 2c_i} \quad (36)$$

holds, where we used (27). By substitution in (17) we obtain an optimization problem in variable $\mathbf{x}_{i,i}$ only, that is

$$\mathbf{x}_{i,i} = \operatorname{argmin}_{\mathbf{x}} \frac{1}{2}\|\mathbf{x} - \mathbf{y}_{i,i}\|^2 + \sum_{j \in \mathcal{N}_i} \frac{\frac{1}{2}([\|\mathbf{x} - \tilde{\mathbf{y}}_{i,j}\| - r_{i,j}]^+)^2}{\tilde{c}_{i,j}N_i} \quad (37)$$

where

$$\tilde{\mathbf{y}}_{i,j} = \begin{cases} \mathbf{y}_{i,j} & j \notin \mathcal{A} \\ \mathbf{a}_j & j \in \mathcal{A} \end{cases}, \quad \tilde{c}_{i,j} = \begin{cases} 2c_i & j \notin \mathcal{A} \\ 1 + 2c_i & j \in \mathcal{A} \end{cases} \quad (38)$$

to separately take into account for the cases where the neighbor j is or is not an anchor. This is a convex problem only for \tilde{F}_i . For large c_i , however, the function tends to $\|\mathbf{x} - \mathbf{y}_{i,i}\|^2$, which is convex by construction in any case.

Because of the very limited dimension of the problem (n in fact is at most equal to 3), the local optimization problem given by (37) is an easy task which can be accomplished by standard optimization methods. In general an algorithm relying on the method of Newton can be chosen to obtain a fast convergence,

in which case we will be using the gradient and Hessian

$$\begin{aligned}\nabla &= \mathbf{x} - \mathbf{y}_{i,i} + \sum_{j \in \mathcal{N}_i} \frac{[\|\mathbf{q}_j\| - r_{i,j}]^\bullet}{\tilde{c}_{i,j} N_i \|\mathbf{q}_j\|} \mathbf{q}_j \\ \nabla^2 &= \mathbf{I}_n \left(1 + \sum_{j \in \mathcal{N}_i} \frac{[\|\mathbf{q}_j\| - r_{i,j}]^\bullet}{\tilde{c}_{i,j} N_i \|\mathbf{q}_j\|} \right) \\ &\quad + \sum_{j \in \mathcal{N}_i} \frac{r_{i,j} \mathbf{q}_j \mathbf{q}_j^T}{\tilde{c}_{i,j} N_i \|\mathbf{q}_j\|^3} 1^\bullet (\|\mathbf{q}_j\| - r_{i,j}),\end{aligned}\quad (39)$$

where $\mathbf{q}_j = \mathbf{x} - \tilde{\mathbf{y}}_{i,j}$, and where we used (38). Note that, because of the very limited dimension n , inversion of the Hessian does not constitute a bottleneck for implementation. Also note that, thanks to the convexity property, the method of Newton leads to an exact result when \bar{F}_i is used. In the transition to the non-convex function F_i , no guarantee is in general available that the global minimum is reached, unless c_i is so large that the problem has become convex (and, incidentally, this latter property guarantees that the distributed algorithm will converge in any case). It is however reasonable to expect that the final outcome will improve over the solution to the convexified problem, as we will discuss in detail in Section VI.

VI. PERFORMANCE EVALUATION AND DISCUSSION

Performance of the proposed method is tested on the networks previously used in [12]. Specifically, these are the $N = 40$ nodes network with $|\mathcal{A}| = 10$ anchors depicted in [12, Fig. 2], and two larger benchmark tests available in Stanford's Computational Optimization Laboratory web site [18], namely a $N = 500$ node network with $|\mathcal{A}| = 10$ anchors, and a $N = 1000$ node network with $|\mathcal{A}| = 20$ anchors. All these networks are assumed to have nodes distributed over a 1×1 square area, so that the coordinate dimension is $n = 2$. For the smaller network of size $N = 40$ noisy distance measurements are generated according to an AWGN model with standard deviation $\sigma = 0.1$ (moderate noise level) and $\sigma = 0.01$ (low noise level). For the two larger networks, the noisy distance measurements given by the benchmark tests were used, which correspond to AWGN noises with standard deviation of, respectively, $\sigma = 0.02$ and $\sigma = 0.007$.

In order to provide a complete insight on the relation of Algorithm 1 with the solutions already available from the literature, the following algorithms are compared:

- 1) SF, namely the *simple and fast* method of [11] implemented via Nesterov's method;
- 2) ADMM-SF, namely the SF method implemented via the ADMM approach of Algorithm 1 where only the convex relaxation \bar{F}_i is used, and the transition to the original non-convex formalization is not activated;
- 3) SDP, namely the SDP algorithm proposed by [10];
- 4) ADMM-NC, namely the non-convex approach proposed in [12];
- 5) ADMM-H, where H stands for *hybrid*, namely the full implementation of Algorithm 1, with an active transition from convex to non-convex functions.

All algorithms are implemented in *MATLAB*, and the minima to local optimization problems (36)-(38) are identified by the *fminunc* solver provided by the *MATLAB Optimization Toolbox*. System performance is evaluated via the root mean squared error (RMSE) measure

$$\text{RMSE} = \sqrt{\frac{1}{N} \sum_{i \in \mathcal{N}} \|\mathbf{x}_{i,i} - \mathbf{p}_i\|^2}, \quad (40)$$

with \mathbf{p}_i the true position and $\mathbf{x}_{i,i}$ the estimate locally available at node i . System parameters are separately optimized for each algorithm, in order to minimize the number of iterations required for convergence to the optimal point. A detailed prospect on the more relevant chosen parameters is given in Table I. To these it must be added that $\rho = \frac{1}{5}$ was selected

TABLE I
SIMULATION SETTINGS

	$N = 40$	$N = 500$	$N = 1000$
ADMM-SF	$\epsilon_c = 0.005$	$\epsilon_c = 0.004$	$\epsilon_c = 0.002$
ADMM-NC	$\zeta_c = 0.1$	$\zeta_c = 0.1$	$\zeta_c = 0.05$
ADMM-H	$\epsilon_c = 0.005$	$\epsilon_c = 0.004$	$\epsilon_c = 0.002$
	$\zeta_c = 0.1$	$\zeta_c = 0.2$	$\zeta_c = 0.05$
	$\tau_c = 0.04$	$\tau_c = 0.06$	$\tau_c = 0.02$

for the SDP algorithm, $\lambda_{\max} = 10^3$ was selected for all the ADMM based algorithms, and $\delta_c = 1.01$ and $\theta_c = 0.98$ were chosen for ADMM-NC and -H. The starting point is set to the all zero vector $\mathbf{x}_{i,i} = \mathbf{0}$, that is, we are investigating a worst case solution where no a priori information is available.

A complete view on RMSE performance for the various algorithmic approaches is given in Fig. 2, where Fig. 2.(a) and (b) refer to the smaller network of $N = 40$ nodes (two different noise levels), while Fig. 2.(c) and (d) refer to the larger networks of, respectively, $N = 500$ and $N = 1000$ nodes. For the smaller network of $N = 40$ nodes the RMSE value was obtained as an average performance over 50 noise realizations, while for the other networks an average operation is already ensured by the fact that we are considering a large number of nodes. Note that the behavior is essentially equivalent in all the considered cases, with the only major distinction that a smaller noise contribution ensures a gain in localization accuracy. The important aspects to be observed are the following.

We first note that, in the comparison between the SF and the ADMM-SF algorithm, the ADMM approach provides a significant advantage in terms of convergence speed. This behavior may seem to contradict the widely known fact that, if t is the iteration number, the ADMM is known to exhibit $O(1/t)$ convergence speed [19], while for Nesterov's method the expected speed is $O(1/t^2)$ [11]. This is only apparently unreasonable since the convergence rate of Nesterov's method is referring to a convergence to the *exact* minimum point, while for ADMM it is capturing the convergence in the dual domain while the convergence to the exact minimum point in the primal domain is known to be exponential [20]. Moreover, it must be observed that in a noisy scenario the RMSE performance rapidly saturates in the vicinity of the

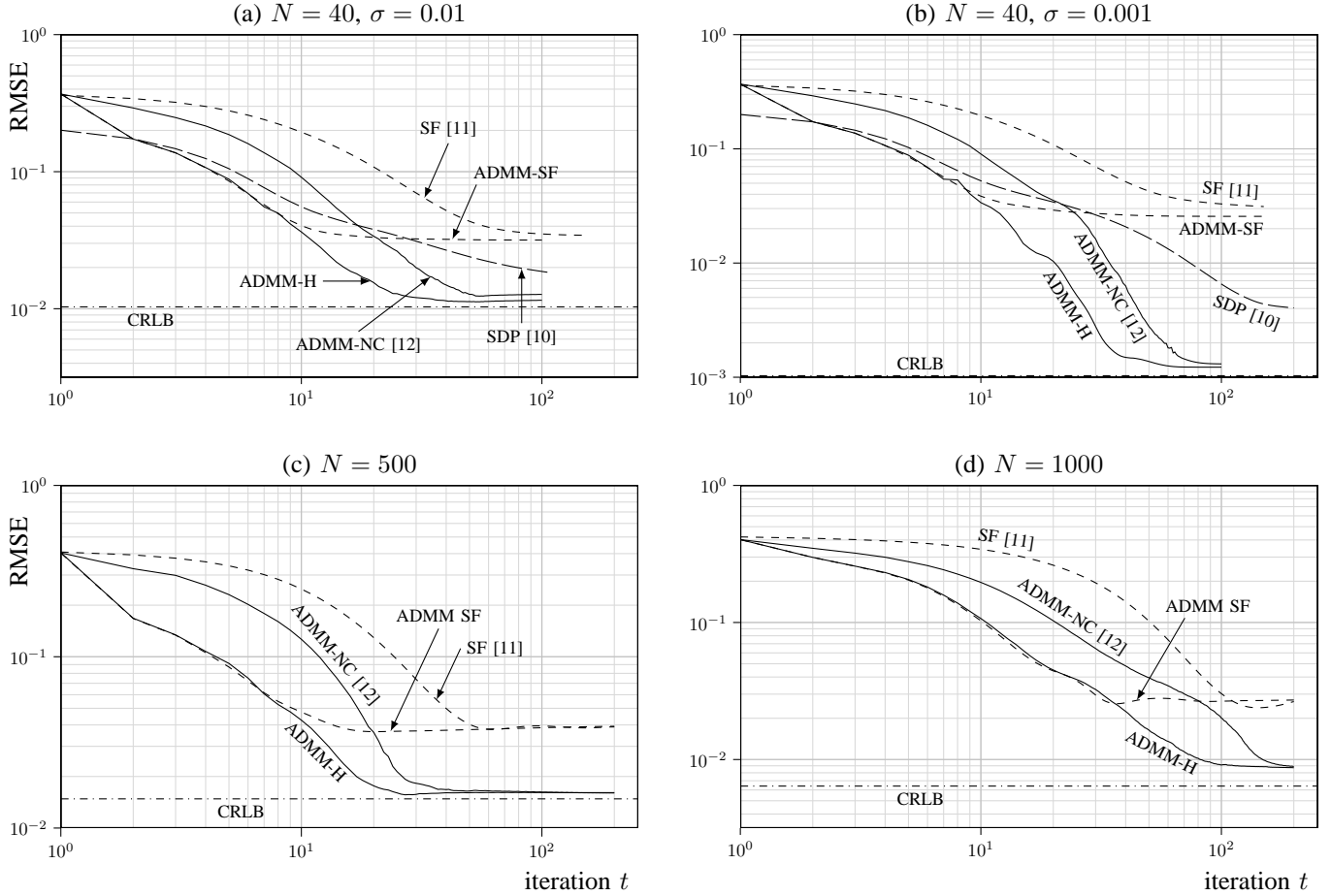


Fig. 2. Performance comparison for networks of size $N = 40$, 500, and 1000.

exact minimum. We can therefore conclude that ADMM is better coordinating the local processing exchange to rapidly move towards the vicinity of the minimum point, and in this sense is to be preferred (this effect is known, e.g., see [15]). We also note that, since the distributed algorithms we are comparing wirelessly exchange information at each iteration, reducing the total number of iterations implies a lower number of communications between nodes, and, therefore, a limitation in energy consumption. A significant saving is also obtained by ADMM-SF in the total time spent for processing, since communication takes a remarkable part of it. The cost to be paid is a more significant local processing effort, which, however, is far below the effort of transmitting and receiving a single packet, which implies a large number of complex operations such as coding, decoding, synchronization, channel estimation, etc. The local computational increase is estimated to be of a factor N_{it} equal to the number of iterations required for the local minimization algorithm (17) (or, better, of its low complexity counterpart (37)) to converge. These are expected to be limited and, as a matter of fact, they were set in simulations to be upper bounded by $N_{it} \leq 3$ (further insights on this issue are given later in Fig. 5).

The second important aspect given in Fig. 2 is the performance comparison between the (different) relaxation methods

used by the SF and the SDP algorithm. It is evident from the figure that SDP is more reliable, which confirms the findings of [11, Fig. 3]. SDP is in any case a very heavy algorithm, and for this reason less suited for implementation (see also [12]). This is also the reason that prevented us from being able to apply SDP to the two larger networks.

We can finally draw some significant conclusions by observing the relation between ADMM based algorithms. Specifically, from Fig. 2 we clearly see that the proposed hybrid approach given by ADMM-H is able to closely follow the rapidly converging behavior of ADMM-SF in its initial iterations, and to successively improve the RMSE performance by setting itself to the (quasi optimal) performance of ADMM-NC. To certify the performance quality, the optimal target given by the Cramer-Rao lower bound (CRLB) (derived according to [21]) is also reported in Fig. 2. With the optimum parameters setting of Fig. 2 the gain in convergence speed between ADMM-H and ADMM-NC is limited to a factor of 2. It is however fair to observe that ADMM-H is much more resilient to the choice of parameters. This aspect is investigated in Fig. 3 which is illustrating that the performance of both ADMM-H and ADMM-SF are only loosely dependent on the parameters choice, i.e., that suboptimal parameters may simply lead to a (slightly) slower convergence. ADMM-NC is instead much

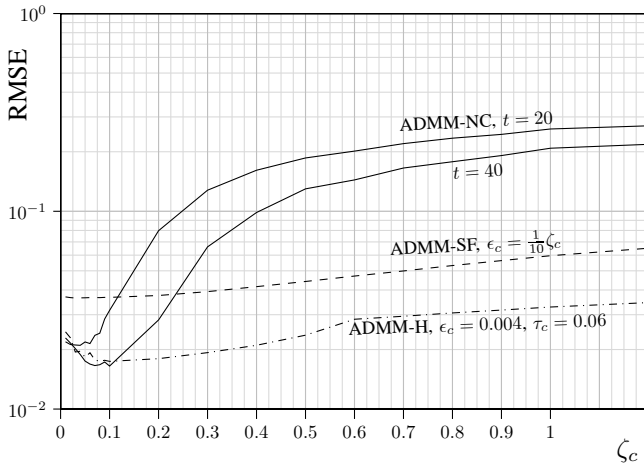


Fig. 3. RMSE performance as a function of parameter ζ_c for the network of size $N = 500$, at iteration $t = 20$ (if not otherwise stated).

more selective, in the sense that parameters choices outside the optimum region may undermine convergence to a good solution or may significantly increase the convergence time. This, in Fig. 3, is true in the region $\zeta_c > 0.25$.

Some further insights on ADMM-H are given in Fig. 4 and Fig. 5. Fig. 4 illustrates the transition between convex and non-convex formulations by showing the percentage of nodes which use the non-convex functions F_i as a function of the iteration number. Note that the transition is much more sudden for the smaller network of $N = 500$ nodes, which in fact requires only 20 iterations to converge. Fig. 5 instead illustrates the computational time spent, per iteration, on both ADMM-H and SF. Both algorithms were implemented in *MATLAB* in a way to make the time calculation fully comparable. Note that the difference is roughly a factor of 4 both in the calculation of the *maximum* and the *average* times (with maximum and average taken with respect to times separately calculated on each node). This is in accordance with the fact that $N_{it} \leq 3$. Although not shown in figure we also observe that, as one can expect, the computational times of ADMM-NC and ADMM-SF (using the efficient formulation (37)-(39)) are essentially equivalent to those of ADMM-H.

A final insight is given in Fig. 6 in a *tracking* context where the network of $N = 500$ nodes is moving and the localization algorithm is applied starting from the solution available from the previous step, except made for the initial localization which starts from an all zero vector. Nodes are assumed to be moving on a random direction, with a velocity taken from a Gaussian distribution, and over a square area. If the square area is assumed to have a 100 m side, then the average velocity is assumed to be 5 km/h (walking speed), the standard deviation is set to 3.33 km/h, and the maximum speed to 15 km/h. The network is built in such a way that nodes within 8.33 m (or 25 m for anchors) are exchanging ranging measurements, and ranging measurements are affected by a standard deviation of 1.66 m (equivalent to that of the original test given in [18]). Nodes are ensured to have access to at least four ranging measurements. Two different algorithms are

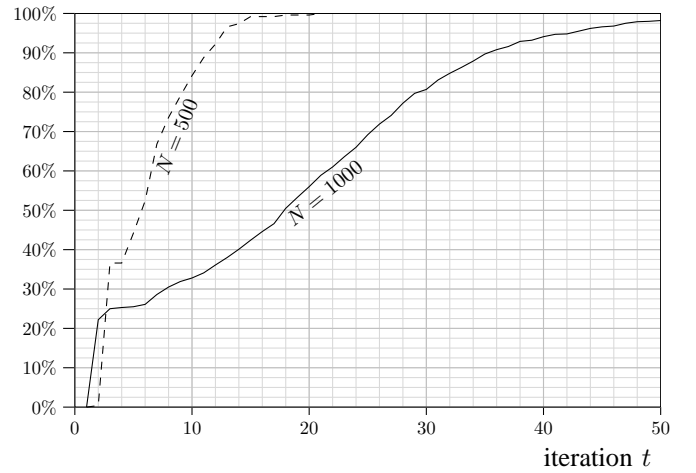


Fig. 4. ADMM-H: percentage of nodes where the local non-convex functional F_i is active, as a function of the iteration number t .

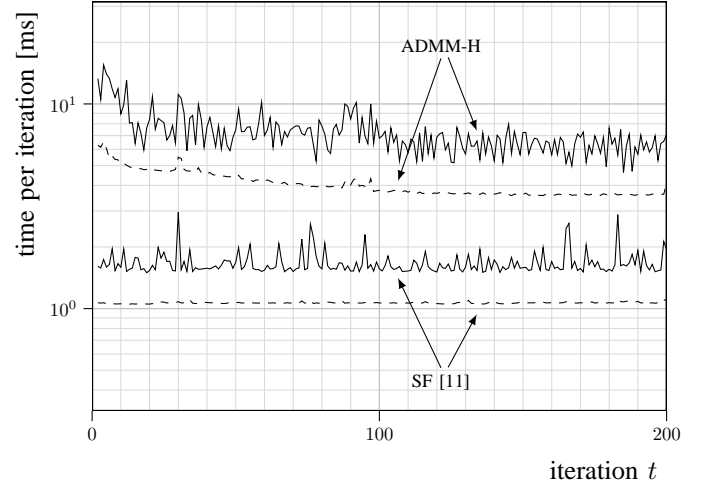


Fig. 5. Computational time per iteration for the network of size $N = 500$: average time per node in dashed lines, and maximum time per node in solid lines. Although not shown, the performance of ADMM-NC and ADMM-SF is comparable to that of ADMM-H.

compared, namely ADMM-H and ADMM-SF which use the parameters of Table I. The ranging and localization measures are updated every second, which corresponds to a step, and each algorithm performs $t = 20$ iterations per step. Note from Fig. 6 that, apart from the very first steps where the algorithms are slowly converging to their respective target performance (this is due to the fact that the maximum number of iterations per step is kept small), then ADMM-H sets itself to a RMSE performance which is approximately four times lower than that of its ADMM-SF counterpart. The optimum level given by the CRLB is also shown to certify the estimation quality.

VII. CONCLUSIONS

In this paper we built upon the SF convex relaxation method proposed in [11] and ameliorated it in two ways. First, by casting the problem into a suitable ADMM formalization, we were able to identify a fully distributed localization algorithm

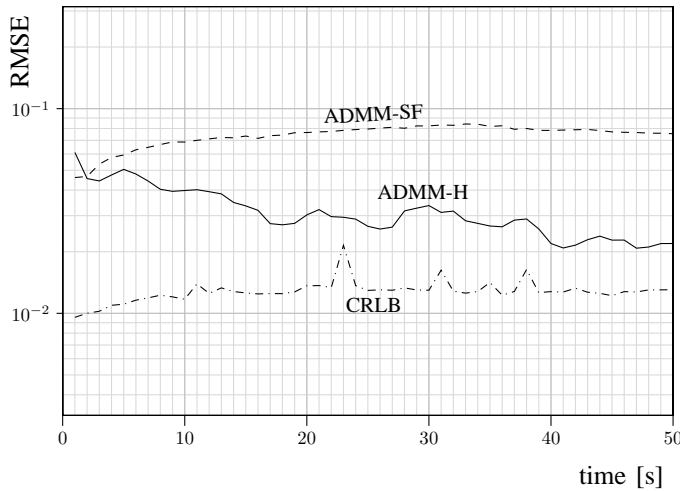


Fig. 6. Algorithm comparison in a tracking scenario with $t = 20$ iterations per localization step. The localization algorithm is run every second on ranging measurements which are corrupted by independent and identically distributed noise samples.

(ADMM-SF) which is scalable, and which sensibly improves in convergence speed over the original proposal based on Nesterov's method. Second, by forcing a transition to the original non-convex function under local convergence, we were able to identify a hybrid algorithm (ADMM-H) with improved localization accuracy. Both algorithms were shown to be robust to parameters choices, to be suited for tracking purposes where the number of iterations per localization step is limited, and to be scalable in that they guarantee fast convergence and accuracy also with large networks. The proposed algorithms were also shown to have limited computational complexity, which guarantees their usability in practical contexts where energy consumption is an issue.

REFERENCES

- [1] J. Yick, B. Mukherjee, and D. Ghosal, "Wireless sensor network survey," *Computer networks*, vol. 52, no. 12, pp. 2292–2330, 2008.
- [2] F. Cadger, K. Curran, J. Santos, and S. Moffett, "A survey of geographical routing in wireless ad-hoc networks," *Communications Surveys & Tutorials*, *IEEE*, vol. 15, no. 2, pp. 621–653, 2013.
- [3] K. Langendoen and N. Reijers, "Distributed localization in wireless sensor networks: a quantitative comparison," *Computer Networks*, vol. 43, no. 4, pp. 499–518, 2003.
- [4] K. W. Cheung and H.-C. So, "A multidimensional scaling framework for mobile location using time-of-arrival measurements," *IEEE Transactions on Signal Processing*, vol. 53, no. 2, pp. 460–470, 2005.
- [5] B. Denis, J.-B. Pierrot, and C. Abou-Rjeily, "Joint distributed synchronization and positioning in UWB ad hoc networks using TOA," *IEEE Transactions on Microwave Theory and Techniques*, vol. 54, no. 4, pp. 1896–1911, 2006.
- [6] G. Mao, B. Fidan, and B. Anderson, "Wireless sensor network localization techniques," *Computer networks*, vol. 51, no. 10, pp. 2529–2553, 2007.
- [7] Q. Shi, C. He, H. Chen, and L. Jiang, "Distributed wireless sensor network localization via sequential greedy optimization algorithm," *IEEE Transactions on Signal Processing*, vol. 58, no. 6, pp. 3328–3340, 2010.
- [8] Z. Wang, S. Zheng, Y. Ye, and S. Boyd, "Further relaxations of the semidefinite programming approach to sensor network localization," *SIAM Journal on Optimization*, vol. 19, no. 2, pp. 655–673, 2008.

- [9] J. Cota-Ruiz, J.-G. Rosiles, P. Rivas-Perea, and E. Sifuentes, "A distributed localization algorithm for wireless sensor networks based on the solutions of spatially-constrained local problems," *IEEE Sensors Journal*, vol. 13, no. 6, pp. 2181–2191, June 2013.
- [10] A. Simonetto and G. Leus, "Distributed maximum likelihood sensor network localization," *IEEE Transactions on Signal Processing*, vol. 62, no. 6, pp. 1424–1437, March 2014.
- [11] C. Soares, J. Xavier, and J. Gomes, "Simple and fast convex relaxation method for cooperative localization in sensor networks using range measurements," *IEEE Transactions on Signal Processing*, vol. 63, no. 17, pp. 4532–4543, Sept 2015.
- [12] T. Erseghe, "A distributed and maximum-likelihood sensor network localization algorithm based upon a nonconvex problem formulation," *IEEE Transactions on Signal and Information Processing over Networks*, vol. 1, no. 4, pp. 247–258, Dec 2015.
- [13] —, "A distributed approach to the OPF problem," *EURASIP Journal on Advances in Signal Processing*, vol. 2015, no. 1, p. 45, 2015.
- [14] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and distributed computation: numerical methods*. Athena Scientific, 1997.
- [15] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, *Distributed optimization and statistical learning via the alternating direction method of multipliers*, ser. Foundations and Trends in Machine Learning. Now Publishers Inc, 2010, vol. 3, no. 1, pp. 1–122.
- [16] R. Andreani, E. G. Birgin, J. M. Martínez, and M. L. Schuverdt, "On augmented Lagrangian methods with general lower-level constraints," *SIAM Journal on Optimization*, vol. 18, no. 4, pp. 1286–1309, 2007.
- [17] E. Birgin and J. Martínez, *Practical Augmented Lagrangian Methods for Constrained Optimization*. Philadelphia, PA: Society for Industrial and Applied Mathematics, 2014.
- [18] Y. Ye. Computational Optimization Laboratory. Stanford University, <http://www.stanford.edu/~yyye/Col.html>.
- [19] T. Goldstein, B. O'Donoghue, S. Setzer, and R. Baraniuk, "Fast alternating direction optimization methods," *SIAM Journal on Imaging Sciences*, vol. 7, no. 3, pp. 1588–1623, 2014.
- [20] F. Iutzeler, P. Bianchi, P. Ciblat, and W. Hachem, "Explicit convergence rate of a distributed alternating direction method of multipliers," *IEEE Transactions on Automatic Control*, vol. 61, no. 4, pp. 892–904, April 2016.
- [21] N. Patwari, J. Ash, S. Kyperountas, A. Hero, R. Moses, and N. Correal, "Locating the nodes: cooperative localization in wireless sensor networks," *IEEE Signal Processing Magazine*, vol. 22, no. 4, pp. 54–69, Jul. 2005.